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Thermal Buckling of Rotating Orthotropic Annular Plates

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Introduction

The literature dealing with thermal buckling of circular plates is, indeed, very scarce. Mansfield[1] considered the buckling of a heated circular plate subjected to a temperature which varies parabolically in its plane. Sarkar[2] dealt with the buckling of a circular plate under inplane compression and subjected to temperature distributions along the radius and through the thickness of the plate. Bogdanoff, et al.,[3] determined whether a given radial temperature gradient will cause lateral buckling of a rotating disk. In this Note the thermal gradient required to cause buckling of a rotating orthotropic annular plate is calculated. The plate is also subjected to an external pull load applied at the outer periphery of the plate. A constant thickness plate is considered; however, the method described in this Note can easily be extended to include plates of variable thickness.

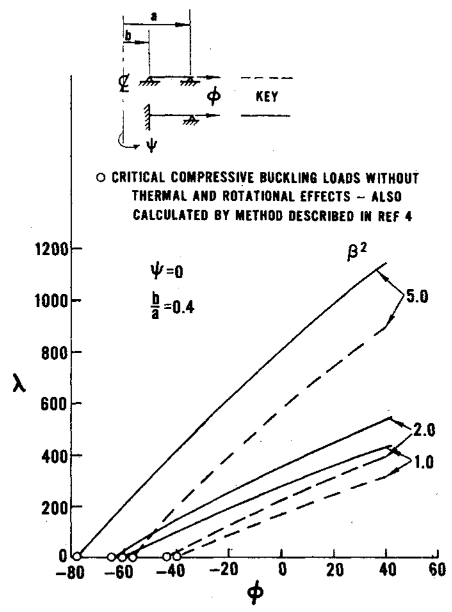

Fig. 1 Thermal buckling parameter vs in-plane loading parameter (neglecting rotational effects).

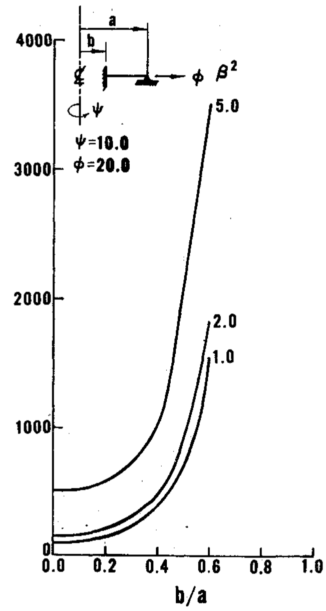

Fig. 2 Thermal buckling parameter vs b/a.

Analysis

The governing axisymmetric equation in terms of the deflection w and the stresses σ_r and σ_θ, including rotational effects, is

$$\Delta w = (h/D_r)(\sigma_r d^2w/dr^2 + \sigma_\theta(1/r)(dw/dr) + \gamma\omega^2 r dw/dr) \quad (1)$$

where

$$\Delta = d^4/dr^4 + (2/r)(d^3/dr^3) - \beta^2/r^2[d^2/dr^2 - (1/r)(d/dr)]$$

r,θ = radial and circumferential coordinates, respectively, $D_r = E_r h^3/12(1 - \nu_{r\theta}\nu_{\theta r})$, $\beta^2 = E_\theta/E_r$, E_θ,E_r = moduli of elasticity in the circumferential and radial directions, respectively, $\nu_{r\theta},\nu_{\theta r}$ = Poisson's ratios, h = plate thickness, γ = mass density, ω = rotational velocity.

The stresses, including a parabolic radial temperature distribution and rotational effects, are given by

$$\sigma_r = C_1 r^{-1+\beta} + C_2 r^{-(1+\beta)} - r^2[(3 + \nu_{r\theta})\gamma\omega^2 + E_\theta\alpha g]/(9 - \beta^2) \quad (2)$$

$$\sigma_\theta = r d\sigma_r/dr + \sigma_r + \gamma\omega^2 r^2$$

where: α = coefficient of thermal expansion assumed to be directionally independent; g = temperature gradient coefficient; $\beta \neq 3$; C_1 and C_2 are determined from the loading conditions at the inner edge ($r = b$) and the outer edge ($r = a$).

Substituting the stresses into Eq. (1) and writing the resulting equation in nondimensional form gives

$$\nabla w - \frac{d^2w}{d\rho^2}(C_3\rho^{-1+\beta} + C_4\rho^{-(1+\beta)} - \rho^2\Psi) - \frac{dw}{d\rho}\left[\frac{\beta}{\rho}(C_3\rho^{-1+\beta} - C_4\rho^{-(1+\beta)}) - 3\rho\Psi\right] - \lambda\left(\rho^2\frac{d^2w}{d\rho^2} + 3\rho\frac{dw}{d\rho}\right) = 0 \quad (3)$$

where: $\nabla = \Delta$ with ρ substituted for r; $C_3 = C_1 a^{1+\beta}h/D_r$; $C_4 = C_2 a^{1-\beta}h/D_r$; $\Psi = (3 + \nu_{r\theta})\gamma\omega^2 h a^4/[D_r(9 - \beta^2)]$; $\lambda = E_\theta\alpha g h a^4/[D_r(9 - \beta^2)]$; $\rho = r/a$.

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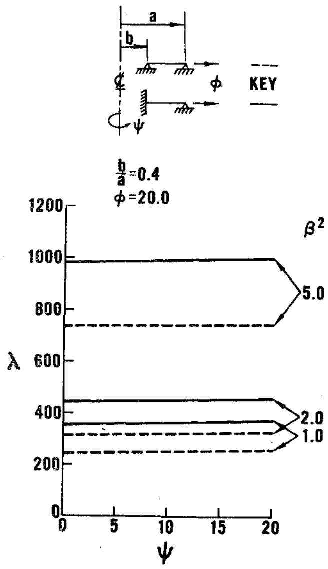

Fig. 3 Thermal buckling parameter vs rotational parameter.

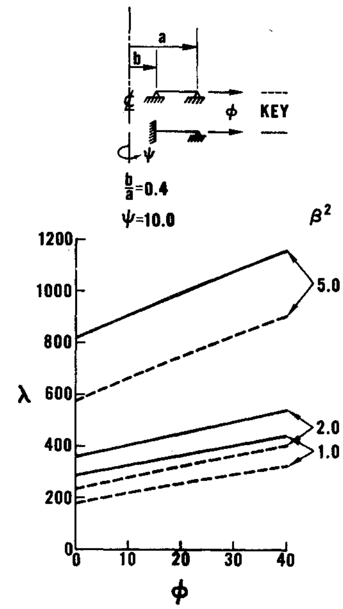

Fig. 4 Thermal buckling parameter vs inplane loading parameter.

The loading conditions, of particular interest to the present investigation are $\sigma_r = 0$ at $r = b$, $\sigma_r = F$ at $r = a$. Using Eq. (2), the constants of integration become $C_3 = (A1 + \lambda A3)/A5$, $C_4 = (A2 + \lambda A4)/A5$, where $A1 = [(\Psi + \phi)c^{-\beta-1} - \Psi c^2)/A5$, $A2 = [\Psi(c^2 - c^{\beta-1}) - \phi c^{\beta-1}]/A5$, $A3 = (c^{-\beta-1} - c^2)/A5$, $A4 = (c^2 - c^{\beta-1})/A5$, $A5 = c^{-\beta-1} - c^{\beta-1}$, $c = b/a$, $\phi = Fa^2h/D_r$.

Nontrivial solutions of Eq. (3) exist for particular values of the eigenvalue λ. The first eigenvalue determines the lowest critical thermal gradient coefficient for incipient buckling.

Method of Solution

The technique employed to solve Eq. (3) closely follows that given in Ref. 4. Writing Eq. (3) in finite-difference form and collecting coefficients of terms with like subscripts gives

$$w_iA_i + w_{i+1}B_i + w_{i+2}C_i + w_{i-1}D_i + w_{i-2}E_i - \lambda(w_iF_i + w_{i+1}G_i + w_{i-1}H_i) = 0 \qquad i = 1 \qquad n \quad (4)$$

where $A_i = 6/t^4 - 2AA_i/t^2$, $B_i = -4/t^4 - 2/t^3\rho_i + AA_i/t^2 + BB_i/2t$, $C_i = 1/t^4 + 1/t^3\rho_i$, $D_i = 4/t^4 + 2/t^3\rho_i + AA_i/t^2 + BB_i/2t$, $E_i = 1/t^4 - 1/t^3\rho_i$, $F_i = 2CC_i/t^2$, $G_i = CC_i/t^2 + DD_i/2t$, $H_i = CC_i/t^2 - DD_i/2t$, $AA_i = -\rho_i^{\beta-1}A1 - \rho_i^{-\beta-1}A2 + \rho_i^2\Psi - \beta^2\rho_i^{-2}$, $BB_i = -\beta(\rho_i^{\beta-2}A1 - \rho_i^{-\beta-2}A2) + 3\rho_i\Psi + \beta\rho_i^{-3}$, $CC_i = \rho_i^{\beta-1}A3 + \rho_i^{-\beta-1}A4 - \rho_i^2$, $DD_i = \beta(\rho_i^{\beta-2}A3 - \rho_i^{-\beta-2}A4) - 3\rho_i$.

Equation (4) can be written in matrix form as

$$[C](w) = (1/\lambda)[I](w)$$

where $[C] = [M]^{-1}[N]$, $[I]$ = identity matrix, $[M]$ = matrix whose elements are $A_i - E_i$, $[N]$ = matrix whose elements are F_i, G_i, and H_i, (w) = column vector of the displacements.

The boundary conditions for the deflection determine the first two and last two rows of matrix $[M]$ and the first and last rows of matrix $[N]$. To determine the first eigenvalue λ the Vianello-Stodola method[5] was employed and the computations involved were performed on a digital computer.

Results

The critical thermal gradients were determined for a simply-supported plate and for a plate fixed at the inner edge and simply-supported at the outer edge. The nondimensional parameters are, λ the thermal gradient coefficient, Ψ the rotational term, ϕ the external load, b/a the ratio of inner to outer radius and β^2 the ratio of the moduli. In all calculations Poisson's ratios, $\nu_{r\theta} = \nu_{\theta r} = 0.3$ is used. Figure 1 gives the thermal gradient coefficients as a function of external load, neglecting the rotational term ($\Psi = 0$). For a constant temperature distribution ($\lambda = 0$) the external loads are shown to agree with the critical buckling loads ($-\phi$) determined by the methods described in Ref. 4. The thermal gradient coefficients as a function of b/a are given in Fig. 2 and illustrate the great increase in thermal gradient required for buckling as the annular plate approaches a ring. Comparing Figs. 3 and 4 shows that the stiffening influence of the rotational velocity (Ψ) is substantially less than the stiffening influence of the external load. Each figure shows the pronounced effect of the orthotropic nature of the material upon the buckling thermal gradients and illustrates the importance of including the orthotropicity of the material in analyses.

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